Shawnee State University S.U.M.S

Individual Mathematics Competition Test Instructions

Name: Key 1 **School:**

1. Silence and put away your cell phone.
2. Make sure to print your name and school at the top of this page.
3. If are in the 11th or 12th grade, you have 45 minutes to complete the exam. If you are in the 9th or 10th grade, you have 55 minutes to complete the exam.
4. Calculators are allowed but are not necessary.
5. All your work on a problem should be in the space provided below the problem on the exam. You may turn in work on additional paper provide you indicate your name and the problem your work is for.
6. Any work on scrap paper that is not to be considered in scoring should be discarded.
7. Show your supporting work, indicating as to how you arrived at your solution. Proper communication of mathematical results counts; clarity is important.
8. Give exact values for your answers. For example, $\sqrt{3}$ is an exact answer. 1.732 is an approximation of $\sqrt{3}$ not an exact value. 1.732050808 is better approximation (more precise) but is still not an exact value.

1) Alicia flips a coin twice. Later all she can remember is that one of the two flips was tails. She doesn't remember the outcome of the other flip. What is the probability that both flips were tails?

Solution: In flipping a coin twice, the sample space is $S = \{(H,H), (H,T), (T,H), (T,T)\}$

However, with her remembering that one of the two coin flips was tails, the question posed is a conditional probability question of P(both flips tails | at least one tail))

This can be calculated by considering the

$$\frac{\text{number of ways of getting two tails in reduced sample space}}{\text{number in the reduced sample space}}$$

The reduced sample space is $S^* = \{(H,T)(T,H)(T,T)\}$

So P(both flips are tails | at least one flip is tails)

$= \frac{1}{3}$

Alternatively: $P((T,T) \mid \text{at least one tail})) =$

$$\frac{P[(T,T) \cap (\text{at least one tail})]}{P(\text{at least one tail})} = \frac{P(T,T)}{P(\text{at least 1 tail})} = \frac{\frac{1}{4}}{\frac{3}{4}}$$

$= \frac{1}{4} \cdot \frac{4}{3} = \frac{1}{3}.$

2) Let $P(n)$ and $S(n)$ denote the product and the sum, respectively, of the digits of the integer n. For example, $P(23) = 6$ and $S(23) = 5$. Suppose N is a two-digit number such that $N = P(N) + S(N)$. What is the units digit of N?

Let $N = ab$ as a two digit number.

Then $N = a(10) + b(1)$.

So $N = (a)(b) + a + b$.

So $10a + b = ab + a + b$.

So $9a = ab$. So $9a - ab = 0$

So $a(9-b) = 0$.

Since N is a two digit number $a \neq 0$.

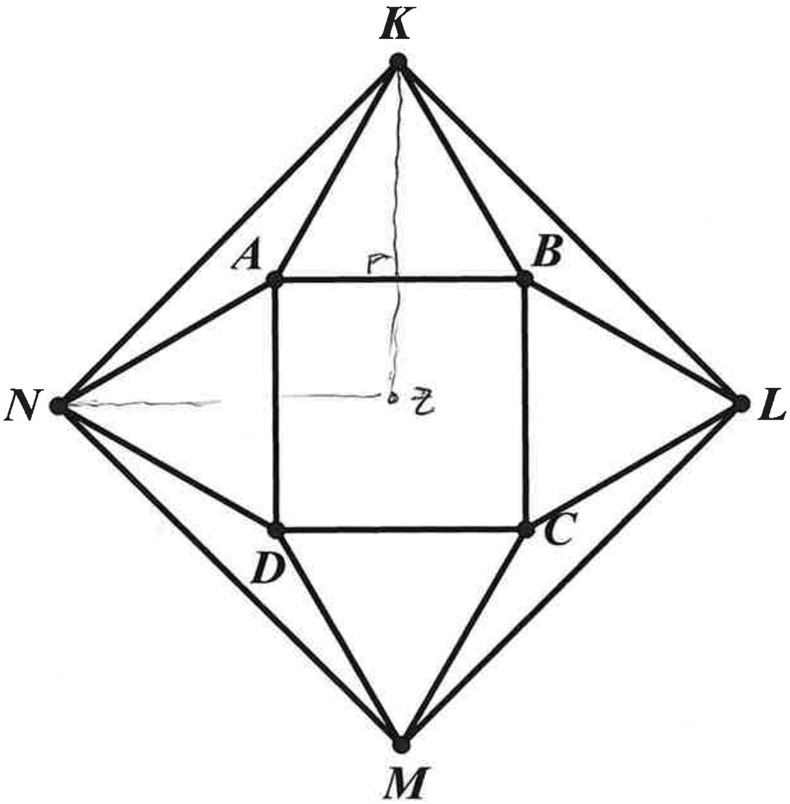
Since $a(9-b) = 0$ and $a \neq 0$

then $9 - b = 0$. So $b = 9$.

and b is the units digit of N.

So the units digit of N is 9.

3) Points K, L, M, and N lie in the plane of the square $ABCD$ so that AKB, BLC, CMD, and DNA are equilateral triangles. The area of $ABCD$ is 16. What is the area of $KLMN$?



Solution: Since $ABCD$ is a square of area 16, then it's sides have length 4. Since AKB is a equilateral triangle its altitude from vertex K to base AB bisects AB. So the altitude using Pythagorean Thm has length $\sqrt{12} = 2\sqrt{3}$. Now extending this altitude down 2 units will have a length of $2 + 2\sqrt{3}$. So the area of the triangle KZN is $\frac{1}{2}(2+2\sqrt{3})(2+2\sqrt{3}) = \frac{1}{2}(4 + 8\sqrt{3} + 12) = 8 + 4\sqrt{3}$.

So the area of $KLMN$ is $4(8+4\sqrt{3}) = 32 + 16\sqrt{3}$

4) Racquetballs are typically sold in cylindrical cans containing 2 balls. If a racquetball has diameter of 2.25 inches and the 2 balls fit exactly into the can so they touch the sides, top, and bottom of the can, how many cubic inches of space in the can is not occupied by balls?

Volume of can: $V_c = \pi r^2 h$. $h = 2.25 + 2.25 = 4.5 = \frac{9}{2}$.

$r = \frac{2.25}{2} = 1.125 = \frac{9}{8}$.

$V_c = \pi \left(\frac{9}{8}\right)^2 \frac{9}{2} = \frac{729}{128}\pi$.

Now $V_B = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{9}{8}\right)^3 = \frac{4}{3}\pi \frac{(81)(9)}{(8)64} = \frac{243}{128}\pi$.

So $2V_B = \frac{486}{128}\pi$

So the remaining volume in can

is $V_c - 2V_B = \frac{729}{128}\pi - \frac{486}{128}\pi = \frac{243\pi}{128}$ in^3.

5) It takes Mary 30 minutes to walk uphill 1 km from her home to school, but it takes her only 10 minutes to walk from school to home along the same route. What is her average speed, in km/hr, for the round trip?

The round trip is a distance of 2 Km. which she completes in 40 minutes, $= \frac{2}{3}$ hr

Since $D = rt$, then her average rate

$$r = \frac{D}{t} = \frac{2\text{ km}}{\frac{2}{3}\text{ hr}} = \frac{6}{2}\text{ km/hr} = 3\text{ km/hr}.$$

6) If $\log(xy^3) = 1$ and $\log(x^2y) = 1$, what is $\log(xy)$?

Since $\log xy^3 = 1$, then $\log x + 3\log y = 1$

Since $\log(x^2y) = 1$, then $2\log x + \log y = 1$.

so $\log x = 1 - 3\log y$. substituting into

the second equation, we get

$$2(1 - 3\log y) + \log y = 1$$

So $2 - 6\log y + \log y = 1$

So $1 = 5\log y$. So $\log y = \frac{1}{5}$.

Since $\log x = 1 - 3\log y$, then $\log x = 1 - \frac{3}{5} = \frac{2}{5}$.

Since $\log(xy) = \log x + \log y$,

then $\log(xy) = \frac{2}{3} + \frac{1}{5} = \frac{3}{5}$.

7) How much bigger than 1 trillion is the first perfect square greater than 1 trillion? Note 1 trillion is equal to 10^{12}.

Since 1 trillion $= 10^{12}$ and $\sqrt{10^{12}} = 10^6$ then 1 trillion is a perfect square that is given by $(10^6)^2$. So the next perfect square is $(10^6+1)^2$

$= 10^{12} + 2(10^6) + 1.$

Note $10^{12} + 2(10^6) + 1 - 10^{12} =$

$2(10^6) + 1 = 2{,}000{,}000 + 1$

$= 2{,}000{,}001.$

8) Find all values of x that satisfy the equation $|x + 6| = 3x + 4$.

Solution: To simplify $|x+6|$ we will look for solutions in each of two of all possible cases.

Case 1: $x \geq -6$.

Then $|x+6| = 3x+4$ means

$$x+6 = 3x+4$$

So $2 = 2x$. So $x = 1$. and $x=1 \geq -6$.

So $x=1$ is a solution

Case 2: Consider x-values, when $x<-6$

So $|x+6| = 3x+4$ means in this case

$-x-6 = 3x+4$

so

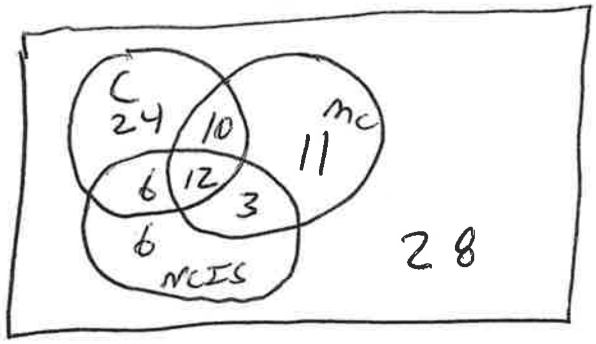
$-10 = 4x$. So $x = -5/2$, but $x = -5/2 \not< -6$

So $x = \frac{-5}{2}$ is not a solution. So

$x=1$ is only solution

9) 100 people are surveyed. It is found that 52 people like "Castle", 36 people like "Major Crimes", and 27 people like "NCIS." In addition, it is found that 22 people like both "Castle" and "Major Crimes", 18 people like both "Castle" and "NCIS", 15 people like both "Major Crimes" and "NCIS", and 12 people like all three shows. How many people in the survey only like "Castle"?

Solution:



So 24 people in the survey only like "Castle".

10) Find the equation of the line through the intersection points of $y = x^2 + x - 2$ and $y = -x^2 + x + 4$.

Solution: The intersection point between the two parabolas is:

$$-x^2 + x + 4 = x^2 + x - 2$$

So $-2x^2 = -6$. So $x^2 = 3$. So $x = \pm\sqrt{3}$.

For $x = \sqrt{3}$, $y = -3 + \sqrt{3} + 4 = 1 + \sqrt{3}$. So one point of intersection is $(\sqrt{3}, 1+\sqrt{3})$

For $x = -\sqrt{3}$, $y = -3 - \sqrt{3} + 4 = 1 - \sqrt{3}$.

So another point of intersection is $(-\sqrt{3}, 1-\sqrt{3})$.

To find the equation of the line through $(\sqrt{3}, 1+\sqrt{3})$ and $(-\sqrt{3}, 1-\sqrt{3})$, note the slope is

$$\frac{(1+\sqrt{3}) - (1-\sqrt{3})}{\sqrt{3} - -\sqrt{3}} = \frac{2\sqrt{3}}{2\sqrt{3}} = 1.$$

So $y - (1+\sqrt{3}) = 1(x - \sqrt{3})$. So $y - 1 - \sqrt{3} = x - \sqrt{3}$.

So $y = x + 1$.